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## A note on regularity criterion for 3D compressible nematic liquid crystal flows

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**Abstract**

In this article, we prove a regularity criterion for the local strong solutions to a simplified hydrodynamic flow modeling the compressible, nematic liquid crystal materials.

**Mathematics Subject Classifications (2010):** 35M10; 76N10; 82D30.

**Keywords:** blow-up criterion, compressible, nematic, liquid crystals

**1 Introduction**

In this article, we consider the following simplified version of Ericksen-Leslie system modeling the hydrodynamic flow of compressible, nematic liquid crystals (see: [1,2])

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u - (\lambda + \mu \nabla \operatorname{div} u) = -\Delta d \cdot \nabla d, \quad (1.2)$$

$$\partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad (1.3)$$

$$(\rho, u, d)(x, 0) = (\rho_0, u_0, d_0)(x), |d_0| = 1, x \in \mathbb{R}^3. \quad (1.4)$$

Here  $\rho$  is the density,  $u$  is the fluid velocity and  $d$  represents the macroscopic average of the nematic liquid crystal orientation field,  $p(\rho) := a\rho^\gamma$  is the pressure with positive constants  $a > 0$  and  $\gamma \geq 1$ .  $\mu$  and  $\lambda$  are the shear viscosity and the bulk viscosity coefficients of the fluid respectively, which are assumed to satisfy the following physical condition:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

(1.1) and (1.2) is the well-known compressible Navier-Stokes system with the external force  $-\Delta d \cdot \nabla d$ . (1.3) is the well-known heat flow of harmonic map when  $u = 0$ .

Very recently, Ericksen [3] proved the following local-in-time well-posedness:

**Proposition 1.1.** *Let  $\rho_0 \in W^{1,q} \cap H^1 \cap L^1$  for some  $q \in (3, 6]$  and  $\rho_0 \geq 0$  in  $\mathbb{R}^3$ ,  $\nabla u_0 \in H^1$ ,  $\nabla d_0 \in H^2$  and  $|d_0| = 1$  in  $\mathbb{R}^3$ . If, in additions, the following compatibility condition*

$$-\mu \Delta u_0 - (\lambda + \mu) \nabla \operatorname{div} u_0 - \nabla p(\rho_0) - \Delta d_0 \cdot d_0 = \sqrt{\rho_0} g \quad \text{for some } g \in L^2(\mathbb{R}^3) \quad (1.5)$$

holds, then there exist  $T_0 > 0$  and a unique strong solution  $(\rho, u, d)$  to the problem (1.1)-(1.4).

Based on the above Proposition 1.1, Huang et al. [4] proved the following regularity criterion:

$$\int_0^T \|\mathcal{D}(u)\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2 dt < \infty, \quad (1.6)$$

where  $\mathcal{D}(u) := \frac{1}{2} (\nabla u + {}^t \nabla u)$ .

The aim of this note is to refine (1.6) as follows.

**Theorem 1.2.** *Let the assumptions in Proposition 1.1 holds true. If*

$$\int_0^T \|\mathcal{D}(u)\|_{L^\infty} + \|\nabla d\|_{BMO}^2 dt < \infty, \quad (1.7)$$

*then the solution  $(\rho, u, d)$  can be extended beyond  $T > 0$ .*

*Here BMO denotes the space of functions of bounded mean oscillations.*

In this note, we will use the following inequality [5]:

$$\|u\|_{L^p} \leq C \|u\|_{BMO}^{1-\frac{q}{p}} \|u\|_{L^q}^{\frac{q}{p}} \quad (1 < q < p < \infty). \quad (1.8)$$

For the standard nematic liquid crystal flows, we refer to recent studies in [6,7].

## 2 Proof of Theorem 1.2

Since  $(\rho, u, d)$  is the local strong solution, we only need to prove

$$\nabla d \in L^2(0, T; L^\infty). \quad (2.1)$$

By the same calculations as that in [4], it is easy to show that

$$\begin{aligned} \|\rho\|_{L^\infty(0, T; L^\infty)} &\leq C, \\ \int_0^T \int \rho |u|^2 + |\nabla d|^2 dx + \int_0^T \int |\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2 dx dt &\leq C. \end{aligned} \quad (2.2)$$

Using (1.8), we see that

$$\begin{aligned} \int_0^T \int |\nabla d|^4 dx dt &\leq C \int_0^T \|\nabla d\|_{BMO}^2 \|\nabla d\|_{L^2}^2 dt \\ &\leq \max_t \|\nabla d\|_{L^2}^2 \int_0^T \|\nabla d\|_{BMO}^2 dt \leq C, \end{aligned}$$

from which and (2.2), we get

$$\int_0^T \int |\Delta d|^2 dx dt \leq 2 \int_0^T \int |\Delta d + |\nabla d|^2 d|^2 dx dt + 2 \int_0^T \int |\nabla d|^4 dx dt \leq C. \quad (2.3)$$

Applying  $\nabla$  to (1.3), testing by  $r|\nabla d|^{r-2}\nabla d$  ( $r \geq 2$ ), using (1.8), we infer that

$$\begin{aligned} & \frac{d}{dt} \int |\nabla d|^r dx + r \int |\nabla d|^{r-2} |\nabla^2 d|^2 + (r-2) |\nabla d|^{r-2} |\nabla |\nabla d||^2 dx \\ &= r \int \nabla (|\nabla d|^2 d) |\nabla d|^{r-2} \nabla d dx - r \int \nabla (u \cdot \nabla d) |\nabla d|^{r-2} \nabla d dx \\ &= r \int |\nabla d|^{r+2} dx - r \int |\nabla d|^{r-2} \nabla_i u^i < \nabla_j d, \nabla_i d > dx - \int u \cdot \nabla |\nabla d|^r dx \\ &= r \int |\nabla d|^{r+2} dx - r \int |\nabla d|^{r-2} \mathcal{D}(u) : \nabla d \otimes \nabla d dx + \int (\operatorname{div} u) |\nabla d|^r dx \\ &\leq C \|\nabla d\|_{\text{BMO}}^2 \|\nabla d\|_{L^r}^r + C \|\mathcal{D}(u)\|_{L^\infty} \|\nabla d\|_{L^r}^r, \end{aligned}$$

which yields

$$\sup_{0 \leq t < T} \|\nabla d\|_{L^r} + \int_0^T \int |\nabla d|^{r-2} |\nabla^2 d|^2 dx dt \leq C. \quad (2.4)$$

Let

$$\dot{f} := f_t + u \cdot \nabla f$$

denotes the material derivative of  $f$ .

Testing (1.2) by  $\dot{u}$ , we derive

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 dx + \int \rho |\dot{u}|^2 dx \\ &= \int < (u \cdot \nabla) u, -\mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u > dx \\ & \quad - \int u \cdot \nabla u \cdot \nabla p(\rho) dx - \int u_t \cdot \nabla p(\rho) dx \\ & \quad - \int u \cdot \nabla u \cdot < \Delta d, \nabla d > dx - \int u_t \cdot < \Delta d, \nabla d > dx \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (2.5)$$

By the same calculations as that in [4], we have

$$\begin{aligned} I_1 &= \mu \int \mathcal{D}(u) : \operatorname{curl} u \otimes \operatorname{curl} u - \frac{1}{2} \operatorname{div} u (\operatorname{curl} u)^2 dx \\ & \quad - (2\mu + \lambda) \int (\nabla u : \nabla u) \operatorname{div} u - \frac{1}{2} (\operatorname{div} u)^3 dx \\ &\leq C \|\mathcal{D}(u)\|_{L^\infty} \|\nabla u\|_{L^2}^2, \\ I_2 &= \int p(\rho) (\nabla u : \nabla u - (\operatorname{div} u)^2) dx - \int u \operatorname{div} u \cdot \nabla p(\rho) dx \\ &\leq C \|\nabla u\|_{L^2}^2 + C \int |\nabla \rho| |u| |\operatorname{div} u| dx \\ &\leq C \|\nabla u\|_{L^2}^2 + C \|u\|_{L^6} \|\nabla \rho\|_{L^2} \|\operatorname{div} u\|_{L^3} \\ &\leq C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^2} \|\mathcal{D}(u)\|_{L^\infty}^{\frac{1}{3}} \|\mathcal{D}(u)\|_{L^2}^{\frac{2}{3}} \\ &\leq C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2 + C \|\mathcal{D}(u)\|_{L^\infty}^{\frac{2}{3}} \|\mathcal{D}(u)\|_{L^2}^{\frac{4}{3}} \\ &\leq C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2 + C (1 + \|\mathcal{D}(u)\|_{L^\infty}) \|\nabla u\|_{L^2}^2, \\ I_3 &= \frac{d}{dt} \int p(\rho) \operatorname{div} u dx - \int \operatorname{div} u \partial_t p(\rho) dx \\ &\leq \frac{d}{dt} \int p(\rho) \operatorname{div} u dx + C + C (1 + \|\mathcal{D}(u)\|_{L^\infty}) \|\nabla u\|_{L^2}^2 \\ & \quad + C \|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2, \\ I_4 &\leq \|u\|_{L^6} \|\nabla u\|_{L^6} \|\Delta d\|_{L^2} \|\nabla d\|_{L^6} \\ &\leq C \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\Delta d\|_{L^2} \text{ (by (2.4) for } r=6) \\ &\leq \varepsilon \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2, \end{aligned}$$

for any  $\epsilon > 0$ .

We denote  $M(d) := \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 I_3$ ,  $I_5$  is estimated as follows, which is slightly different from that in [4]:

$$\begin{aligned} I_5 &= \frac{d}{dt} \int M(d) : \nabla u dx - \int \partial_t M(d) : \nabla u dx \\ &\leq \frac{d}{dt} \int M(d) : \nabla u dx + C \int |\nabla d_t| \|\nabla d\| \|\nabla u\| dx \\ &\leq \frac{d}{dt} \int M(d) : \nabla u dx + C \|\nabla d\|_{L^6} \|\nabla u\|_{L^3} \|\nabla d_t\|_{L^2} \\ &\leq \frac{d}{dt} \int M(d) : \nabla u dx + C \|\nabla u\|_{L^3} \|\nabla d_t\|_{L^2} \quad (\text{by (2.4) for } r = 6) \\ &\leq \frac{d}{dt} \int M(d) : \nabla u dx + \epsilon \|\nabla d_t\|_{L^2}^2 + \epsilon \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2. \end{aligned}$$

Substituting the above estimates into (2.5), we deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \mu + |\nabla u|^2 (\lambda + \mu) |\operatorname{div} u|^2 dx + \int \rho |\dot{u}|^2 dx \\ &\leq C \|\nabla u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2 + C (1 + \|\mathcal{D}(u)\|_{L^\infty}) \|\nabla u\|_{L^2}^2 \\ &\quad + C + \frac{d}{dt} \int p(\rho) \operatorname{div} u + M(d) : \nabla u dx \\ &\quad + C \|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2 + 2\epsilon (\|\nabla d_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) \end{aligned} \quad (2.6)$$

for any  $0 < \epsilon < 1$ .

By the same calculations as that in [4], we write

$$\frac{d}{dt} \|\nabla \rho\|_{L^2}^2 \leq C (1 + \|\mathcal{D}(u)\|_{L^\infty}) \|\nabla \rho\|_{L^2}^2 + \epsilon \|\Delta u\|_{L^2}^2. \quad (2.7)$$

Testing (1.3) by  $\Delta d_t$ , using (2.4), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\Delta d|^2 dx + \int |\nabla d_t|^2 dx = \int (u \cdot \nabla d - |\nabla d|^2 d) \Delta d_t dx \\ &= - \int \nabla (u \cdot \nabla d - |\nabla d|^2 d) \cdot \nabla d_t dx \\ &\leq (\|\nabla u\|_{L^3} \|\nabla d\|_{L^6} + \|u\|_{L^6} \|\Delta d\|_{L^3} + \|\nabla d\|_{L^6}^3 + \|\nabla d\|_{L^6} \|\Delta d\|_{L^3}) \|\nabla d_t\|_{L^2} \\ &\leq C (\|\nabla u\|_{L^3} + \|\nabla u\|_{L^2} \|\Delta d\|_{L^3} + 1 + \|\Delta d\|_{L^3}) \|\nabla d_t\|_{L^2} \\ &\leq C \left( \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} + \|\nabla u\|_{L^2} \|\nabla d\|_{L^6}^{1/2} \|\nabla \Delta d\|_{L^2}^{1/2} + 1 + \|\nabla d\|_{L^6}^{1/2} \|\nabla \Delta d\|_{L^2}^{1/2} \|\nabla d_t\|_{L^2} \right) \\ &\leq C \left( \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} + \|\nabla u\|_{L^2} \|\nabla \Delta d\|_{L^2}^{1/2} + 1 + \|\nabla \Delta d\|_{L^2}^{1/2} \|\nabla d_t\|_{L^2} \right) \\ &\leq \epsilon \|\nabla d_t\|_{L^2}^2 + \epsilon \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 + C + \epsilon \|\nabla \Delta d\|_{L^2}^2. \end{aligned} \quad (2.8)$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\Delta d\|_{L^3}^2 \leq C \|\nabla d\|_{L^6} \|\nabla \Delta d\|_{L^3}. \quad (2.9)$$

Using (1.3), (2.4), and (2.9), we have

$$\begin{aligned} \|\nabla \Delta d\|_{L^2} &\leq \|\nabla d_t + \nabla (u \cdot \nabla d) - \nabla (|\nabla d|^2 d)\|_{L^2} \\ &\leq C \|\nabla d_t\|_{L^2} + C \|u\|_{L^6} \|\Delta d\|_{L^3} + C \|\nabla u\|_{L^3} \|\nabla d\|_{L^6} \\ &\quad + C \|\nabla d\|_{L^6}^3 + C \|\nabla d\|_{L^6} \|\Delta d\|_{L^3} \\ &\leq C \|\nabla d_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla \Delta d\|_{L^3}^{1/2} + C \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} \\ &\quad + C + C \|\nabla \Delta d\|_{L^2}^{1/2}, \end{aligned}$$

whence

$$\|\nabla \Delta d\|_{L^2} \leq C \|\nabla d_t\|_{L^2} + C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2} + C. \quad (2.10)$$

On the other hand, it follows from (1.2), (2.4), and (2.10) that

$$\begin{aligned} \|\Delta u\|_{L^2} &\leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla p(\rho)\|_{L^2} + C \|\Delta d \cdot \nabla d\|_{L^2} \\ &\leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla \rho\|_{L^2} + C \|\nabla d\|_{L^6} \|\Delta d\|_{L^3} \\ &\leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla \rho\|_{L^2} + C \|\Delta d\|_{L^3} \\ &\leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla \rho\|_{L^2} + C \|\Delta d\|_{L^2} + C \|\nabla \Delta d\|_{L^2} \\ &\leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla \rho\|_{L^2} + C \|\Delta d\|_{L^2} + C \|\nabla d_t\|_{L^2} \\ &\quad + C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2} + \frac{1}{2} \|\Delta u\|_{L^2}^2 + C, \end{aligned}$$

which implies

$$\|\Delta u\|_{L^2} \leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla \rho\|_{L^2} + C \|\Delta d\|_{L^2} + C \|\nabla d_t\|_{L^2} + C \|\nabla u\|_{L^2}^2 + C. \quad (2.11)$$

Combining (2.6), (2.7), (2.8), (2.10), and (2.11), taking  $\epsilon$  small enough, using the Gron-wall inequality, we arrive at

$$\nabla d \in L^2(0, T; H^2),$$

whence (2.1) holds true.

This completes the proof.

#### Acknowledgements

The authors would like to thank the referees for careful reading and helpful comments. This article is supported by NSFC (No. 11171154).

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#### Authors' contributions

XC wrote the manuscript and did partial computation. JF proposed the problem and did the main estimates. All authors read and approved the final manuscript.

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Both X. Chen and J. Fan are professors. J. Fan has published more than 90 scientific papers on nonlinear partial differential equations.

#### Competing interests

The authors declare that they have no competing interests.

Received: 11 November 2011 Accepted: 8 March 2012 Published: 8 March 2012

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doi:10.1186/1029-242X-2012-59

**Cite this article as:** Chen and Fan: A note on regularity criterion for 3D compressible nematic liquid crystal flows. *Journal of Inequalities and Applications* 2012 **2012**:59.